1. (a) **(2 marks)** Compute the derivatives of the following functions.

i. $f(x) = e^{1+x^2}\ln(1+x^2)$

$$f'(x) = 2xe^{1+x^2}\cdot\ln(1+x^2) + e^{1+x^2}\cdot\frac{2x}{1+x^2}$$

$$= \boxed{2xe^{1+x^2}\left[\ln(1+x^2) + \frac{1}{1+x^2}\right]}$$

ii. $g(t) = \arcsin(1-t^2)$

$$g'(t) = \frac{-2t}{\sqrt{1-(1-t^2)^2}} = \frac{-2t}{\sqrt{1-(1-2t^2+t^4)}}$$

$$= \frac{-2t}{\sqrt{2t^2-t^4}} = \frac{-2\cancel{t}}{\cancel{t}\sqrt{2-t^2}} = \boxed{\frac{-2}{\sqrt{2-t^2}}}$$

(b) **(2 marks)** Use logarithmic differentiation to find the derivative of

$$y = (\tan x)^{1/x}.$$

$$\ln y = \ln(\tan x)^{1/x} = \frac{1}{x}\ln(\tan x)$$

$$\frac{y'}{y} = -\frac{1}{x^2}\ln(\tan x) + \frac{1}{x}\frac{\sec^2 x}{\tan x}$$

$$y' = y\left[-\frac{1}{x^2}\ln(\tan x) + \frac{1}{x}\frac{\sec^2 x}{\tan x}\right]$$

$$\boxed{y' = (\tan x)^{1/x}\left[\frac{\sec^2 x}{x\tan x} - \frac{\ln(\tan x)}{x^2}\right]}.$$

2. (a) **(2 marks)** Find the absolute maximum and minimum of $f(x) = t\sqrt{4-t^2}$ on $[-1, 2]$.

$$f'(t) = \sqrt{4-t^2} + t \cdot \frac{-\cancel{2}t}{\cancel{2}\sqrt{4-t^2}} = \frac{4-2t^2}{\sqrt{4-t^2}}$$

$f'(t) = 0 \iff 4 = 2t^2 \iff t = \sqrt{2}, \cancel{-\sqrt{2}}$ $(-\sqrt{2} \notin [-1,2])$

$f'(t)$ does not exist @ $t = 2, \cancel{-2}$ $(-2 \notin [-1,2])$

Critical #s: $\sqrt{2}, 2$

End points: $-1, 2$

$f(-1) = -\sqrt{3}$

$f(\sqrt{2}) = \sqrt{2}(\sqrt{4-2}) = 2$

$f(2) = 0$

$\boxed{f(-1) = -\sqrt{3} \text{ is the absolute minimum. } f(\sqrt{2}) = 2 \text{ is the absolute maximum.}}$

(b) **(2 marks)** Find the slope of the tangent line to the curve $2\sin x \cos y = 1$ at the point $(\frac{\pi}{2}, \frac{\pi}{3})$.

$$2\sin x \cos y = 1$$

$$\Rightarrow \sin x \cos y = \frac{1}{2}$$

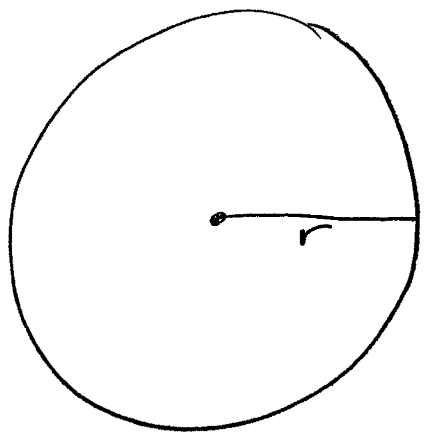
$$\Rightarrow \cos x \cos y - \sin x \sin y \, y' = 0$$

$$y' = \frac{\cos x \cos y}{\sin x \sin y}$$

The slope of the tangent at $(\frac{\pi}{2}, \frac{\pi}{3})$ is

$$y'(\tfrac{\pi}{2}, \tfrac{\pi}{3}) = \frac{\cos\frac{\pi}{2}\cos\frac{\pi}{3}}{\sin\frac{\pi}{2}\sin\frac{\pi}{3}} = \frac{0(\frac{1}{2})}{1(\frac{\sqrt{3}}{2})} = \boxed{0}$$

3. **(3 marks)** Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 2 m/s, how fast is the area of the spill increasing when the radius is 25 m?



Let A, r be the area and radius of the oil spill at time t, respectively.

Then $A = \pi r^2$.

Given $\frac{dr}{dt} = 2$ m/s, find $\frac{dA}{dt}$ when $r = 25$.

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

when $r = 25$,

$$\frac{dA}{dt} = 2\pi(25)(2) = \boxed{100\pi \ m^2/s}$$

4. **(4 marks)** Evaluate the following limits.

(a) $\lim_{x\to 1} \frac{1 - x + \ln x}{1 + \cos \pi x}$ $\left(= \frac{1-1+0}{1-1} = \frac{0}{0}\right)$

$$\overset{H}{=} \lim_{x\to 1} \frac{-1 + \frac{1}{x}}{-\pi \sin \pi x} \left(= \frac{0}{0}\right)$$

$$\overset{H}{=} \lim_{x\to 1} \frac{-\frac{1}{x^2}}{-\pi^2 \cos \pi x}$$

$$= \lim_{x\to 1} \frac{1}{x^2 \pi^2 \cos \pi x} = \boxed{-\frac{1}{\pi^2}}$$

(b) $\lim_{x\to 0^+} \sqrt{x} \ln x$ $= 0 \cdot (-\infty)$

$$= \lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}} \left(= \frac{-\infty}{\infty}\right)$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{-\frac{1}{2}x^{-3/2}}$$

$$= \lim_{x\to 0^+} -2\sqrt{x} = \boxed{0}$$

5. **(5 marks)**

Consider the function

$$y = f(x) = \frac{x^2}{x^2 - 1}.$$

(a) Find the y-intercept and the x-intercept(s) of f.

y-intercept is at $\boxed{y = f(0) = 0.}$

x-intercept? $f(x) = 0 \iff \boxed{x = 0}$

(b) State the domain of f.

domain $= \{x \mid x \neq \pm 1\}$

(c) Determine the equation(s) of any vertical asymptotes and the x-value(s) of any point discontinuities of f, if they exist.

$\lim_{x \to 1} \frac{x^2}{x^2 - 1} = \frac{1}{0}$ does not exist.

$\lim_{x \to -1} \frac{x^2}{x^2 - 1} = \frac{1}{0}$ does not exist.

$\therefore$ f has vertical asymptotes at $\boxed{x = 1 \text{ and } x = -1}$ and f has no point discontinuities.

(d) Determine the equation of any horizontal asymptote of f, if one exists.

$\lim_{x \to \infty} \frac{x^2}{x^2 - 1} = \lim_{x \to -\infty} \frac{x^2}{x^2 - 1} = 1$

$\therefore$ f has a Horizontal Asymptote at $\boxed{y = 1}$.

$$y = f(x) = \frac{x^2}{x^2-1}$$

(e) Find the coordinates of any local maximum and/or local minimum point(s) of f, and state the intervals on which f is increasing and decreasing.

$$f'(x) = \frac{(x^2-1)2x - 2x(x^2)}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$$

$f'(x)=0 \Leftrightarrow x=0$.

$f'(x)$ DNE $\Leftrightarrow x = \pm 1$

f': -1, 0, 1

$+$ V.A. $+$ $-$ V.A. $-$

inc inc dec dec

Local Max

$$\boxed{\begin{array}{l} f \text{ is increasing on } (-\infty,-1)\cup(-1,0). \\ f \text{ is decreasing on } (0,1)\cup(1,\infty). \\ f \text{ has a local maximum @ } (0,0). \end{array}}$$

(f) Find the coordinates of the inflection point(s) of f, if any, and state the intervals of concavity of f.

$$f''(x) = \frac{(x^2-1)^2(-2) - (-2x)[2(x^2-1)2x]}{(x^2-1)^4}$$

$$= \frac{-2\cancel{(x^2-1)}[(x^2-1)-4x^2]}{(x^2-1)^{\cancel{4}\,3}}$$

$$= \frac{2(1+3x^2)}{(x^2-1)^3}.$$

$f''(x)=0$ has no solution, so there are no inflection points.

f is concave down on $(-1,1)$

f is concave up on $(-\infty,-1)\cup(1,\infty)$.

f'': -1, 1

$+$ V.A. $-$ V.A. $+$

cc up cc down cc up

$$y = f(x) = \frac{x^2}{x^2-1}$$

(g) Sketch the graph of the function $y = f(x)$. *Label the important features.*